UNIVERSITY OF LEICESTER

School of Engineering

Bar Elements

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EG3111 – Finite Element Analysis and Design

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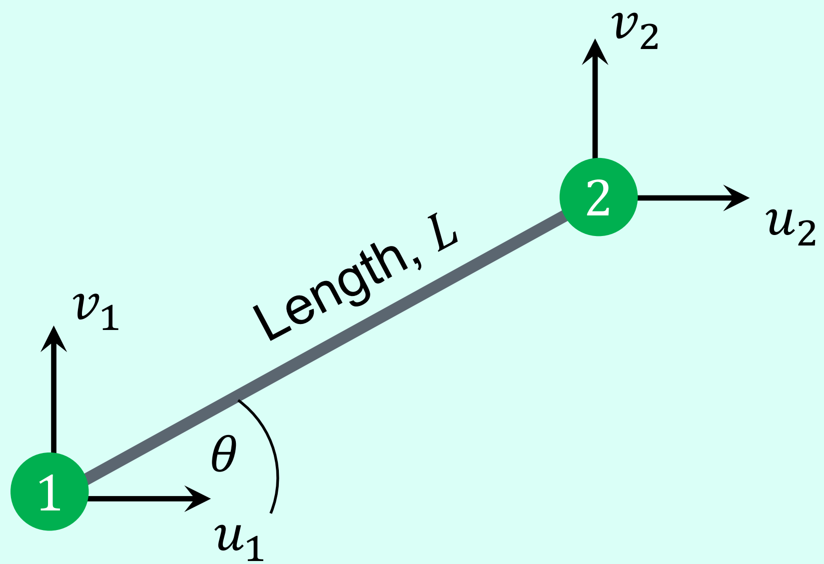
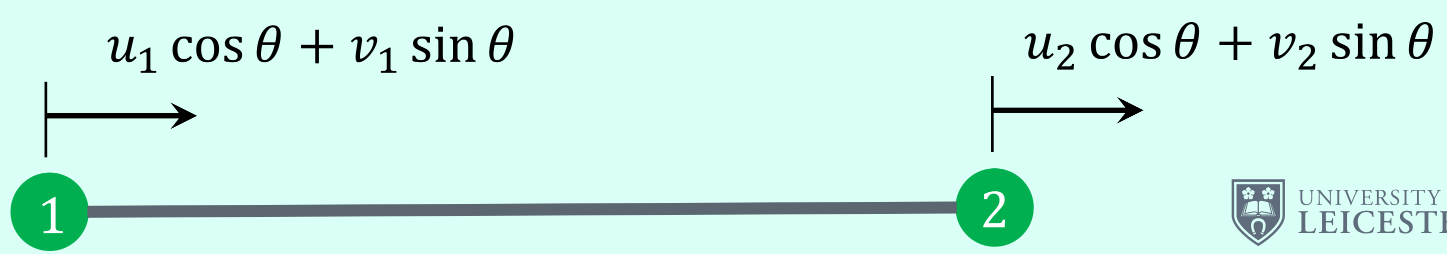
3c. Bar elements for 2D frameworks

Consider a bar element with orientation θ. In 2D the horizontal and vertical displacements are u and v.

DOF



Only displacements parallel to bar axis cause extension/compression



3c. Bar elements for 2D frameworks

Elemental strain energy

$$U^e = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}$$

$$\underline{d^e} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$[k^e] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{d^e} = \begin{bmatrix} u_1\cos\theta + v_1\sin\theta \\ u_2\cos\theta + v_2\sin\theta \end{bmatrix}$$

$$U^e = \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix}.\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$U^e = \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix}.\frac{EA}{L}\begin{bmatrix} u_1 & -u_2 \\ -u_1 & u_2 \end{bmatrix}$$

$$U^e = \frac{1}{2}\frac{EA}{L}[u_1(u_1 - u_2) + u_2(-u_1 + u_2)]$$

$$U^e = \frac{1}{2}\frac{EA}{L}(u_2 - u_1)^2$$

3c. Bar elements for 2D frameworks

$$U^e = \frac{1}{2}\frac{EA}{L}(u_2 - u_1)^2$$

$$U^e = \frac{1}{2}\frac{EA}{L}(u_2\cos\theta + v_2\sin\theta - u_1\cos\theta - v_1\sin\theta)^2$$

$$U^e = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}$$

We wish to consider the four nodal DOF separately.
It is easy to show that U^e is the same with

$$\begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array}$$

$$[k^e] = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \qquad \underline{d^e} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

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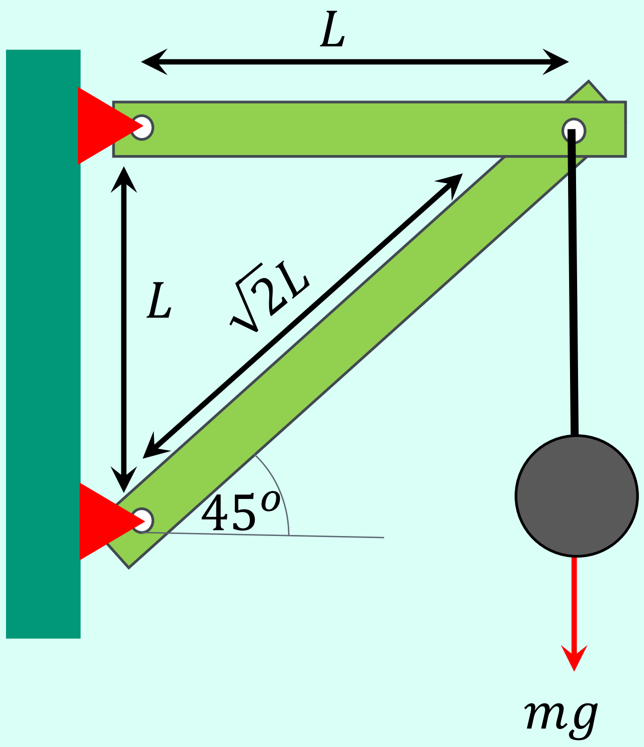
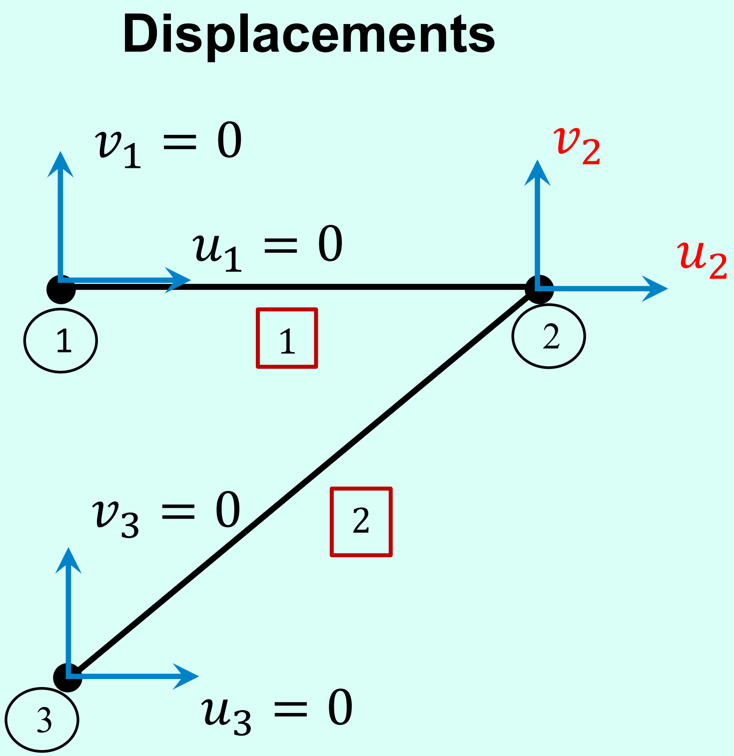
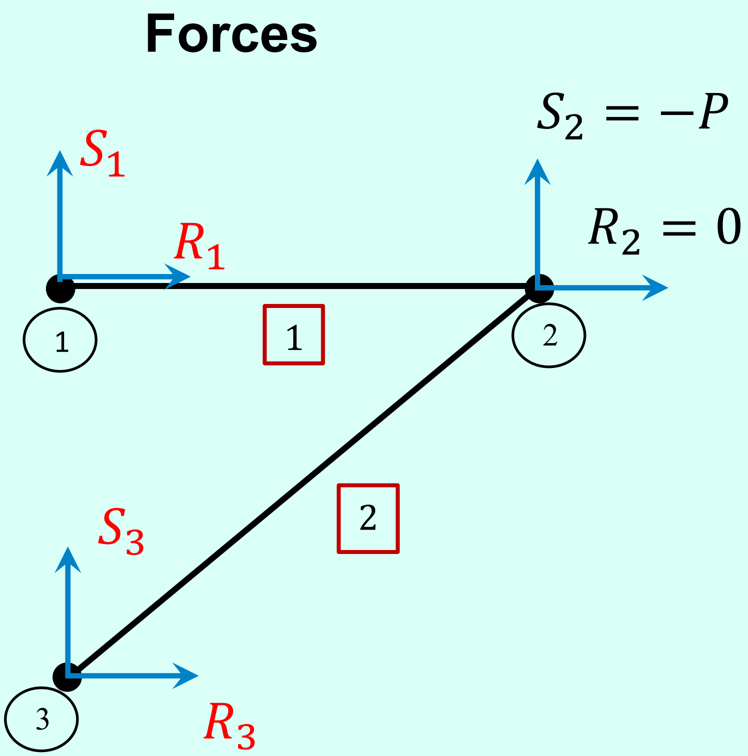
3c. Framework Example

Using two linear bar elements, find the unknown nodal displacements and forces, assuming E and A same for both bars



3c. Framework Example





$$P = mg$$

Unknown DOF in red.

3c. Framework Example

Element (1)

$$\underline{d^{(1)}} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \end{bmatrix}$$

Element 1 $(\boldsymbol{\theta = 0^0})$ $\quad \cos^2 0^o = 1 \quad \cos 0^o \sin 0^o = \sin^2 0^o = 0$

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad k_1 = \frac{EA}{L}$$

3c. Framework Example

Element (2)

$$\underline{d}^{(2)} = \begin{bmatrix} u_3 \\ v_3 \\ u_2 \\ v_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} R_3 \\ S_3 \\ R_2 \\ S_2 \end{bmatrix}$$

Element 2 $(\boldsymbol{\theta = 45^o})$ $\quad \cos^2 45^o = \cos 45^o \sin 45^o = \sin^2 45^o = \frac{1}{2}$

$$\left[k^{(2)}\right] = k_2 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad k_2 = \frac{EA}{2\sqrt{2}L}$$

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3c. Framework Example

Assemble global matrices

$$[k^{(1)}] = k_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \quad (\text{columns: } u_1, v_1, u_2, v_2)$$

$$[k^{(2)}] = k_2 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{matrix} u_3 \\ v_3 \\ u_2 \\ v_2 \end{matrix} \quad (\text{columns: } u_3, v_3, u_2, v_2)$$

$$k_1 = \frac{EA}{L}$$

$$k_2 = \frac{EA}{2\sqrt{2}L}$$

$$\underline{d} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \quad (\text{columns: } u_1, v_1, u_2, v_2, u_3, v_3)$$

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3c. Framework Example

Assemble global matrices

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \end{bmatrix}$$

$$\underline{d} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \\ R_3 \\ S_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ S_1 \\ 0 \\ -P \\ R_3 \\ S_3 \end{bmatrix}$$

3c. Framework Example

Solution

$$[K].\underline{d} = \underline{f} \quad \Longrightarrow \quad \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ S_1 \\ 0 \\ -P \\ R_3 \\ S_3 \end{bmatrix}$$

Partition out 3rd and 4th equations where forces are known.

$$\begin{bmatrix} k_1 + k_2 & k_2 \\ k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

3c. Framework Example

Solution

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \frac{1}{(k_1 + k_2)k_2 - {k_2}^2} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \frac{1}{k_1 k_2} \begin{bmatrix} k_2 P \\ -(k_1 + k_2)P \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = P \begin{bmatrix} \dfrac{1}{k_1} \\ -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) \end{bmatrix}$$

3c. Framework Example

Solution

Substitute known displacements into remaining equations to find unknown forces.

Node 1

$$\begin{bmatrix} -k_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ S_1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} R_1 \\ S_1 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & 0 \end{bmatrix} . P \begin{bmatrix} \dfrac{1}{k_1} \\ -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) \end{bmatrix} = P \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Node 3

$$\begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_3 \\ S_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} R_3 \\ S_3 \end{bmatrix} = \begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix} . P \begin{bmatrix} \dfrac{1}{k_1} \\ -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) \end{bmatrix} = P \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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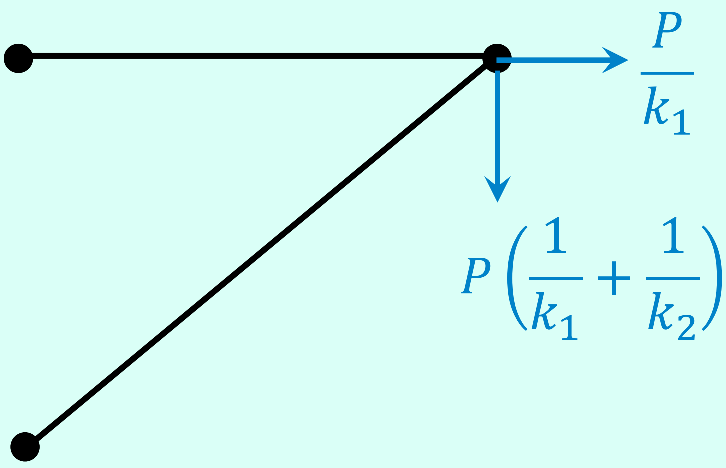
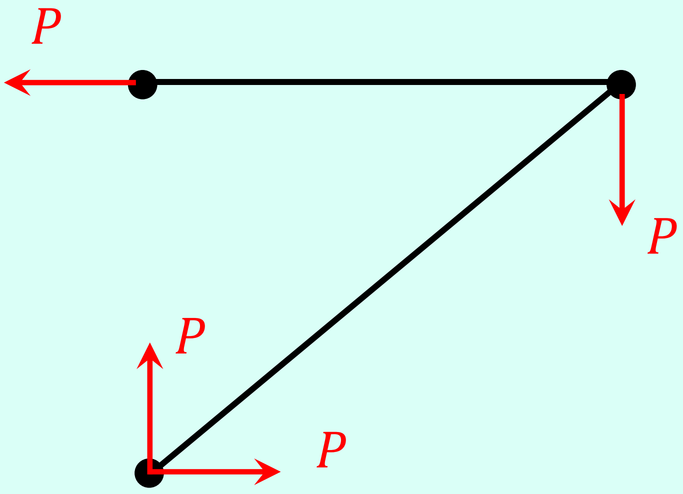
3c. Framework Example

Displacements

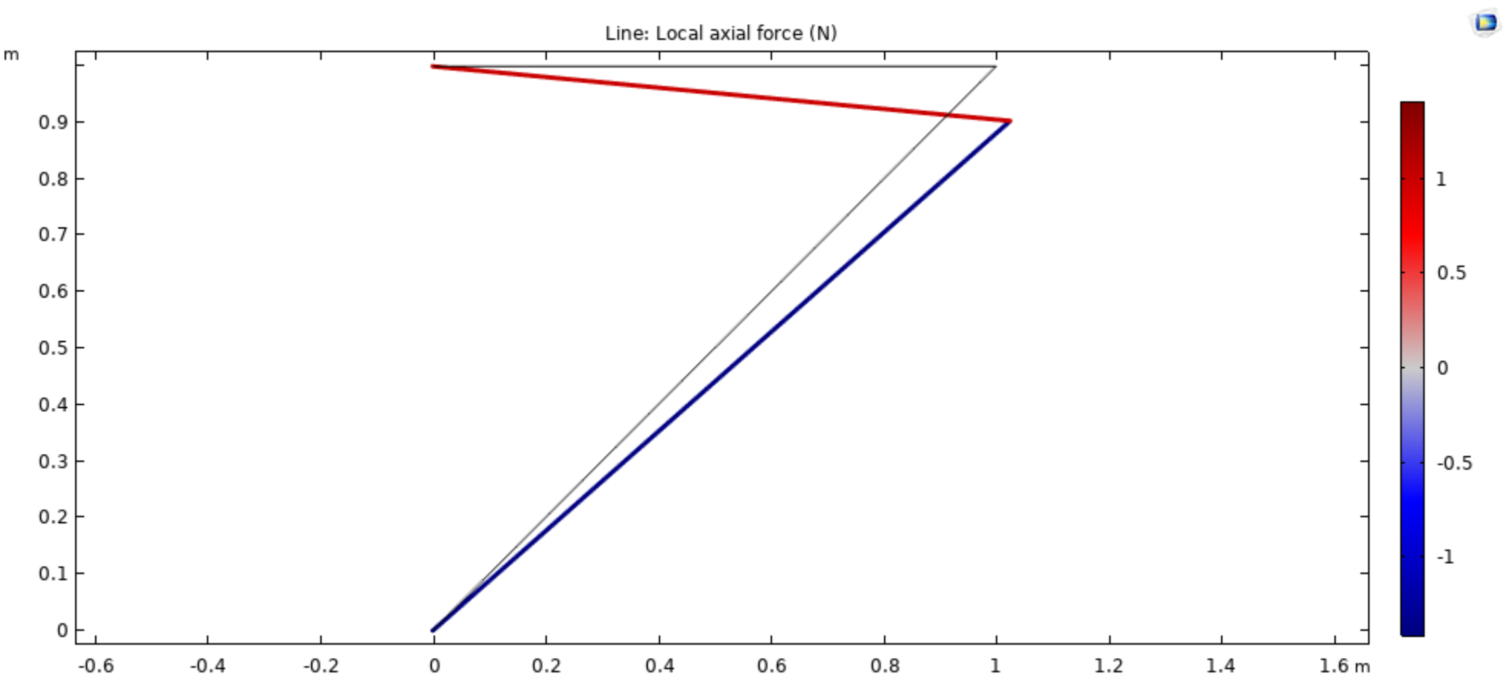


Forces



3c. COMSOL Practical #1

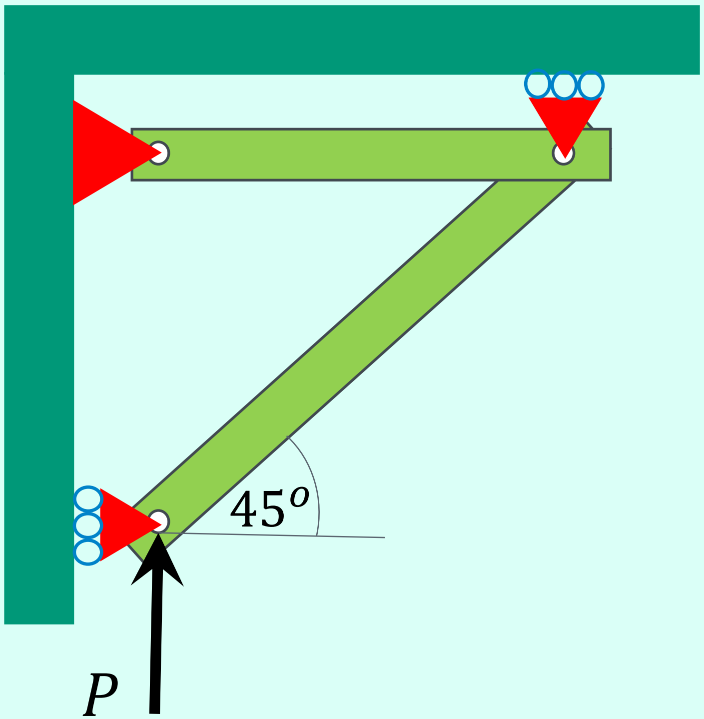
(1) Solve this example and check solution is the same

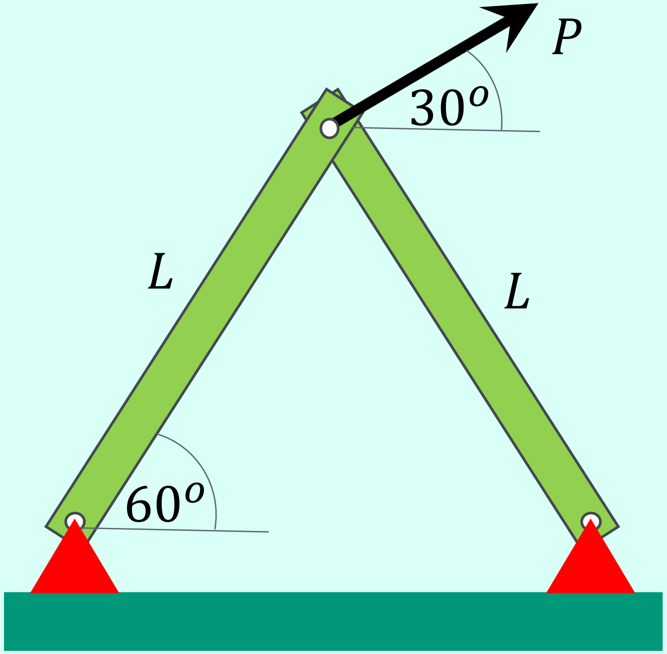


(1) Explore the effects of changing the angle from 45° using a parametric analysis.

3c. COMSOL Practical #1 (continued)

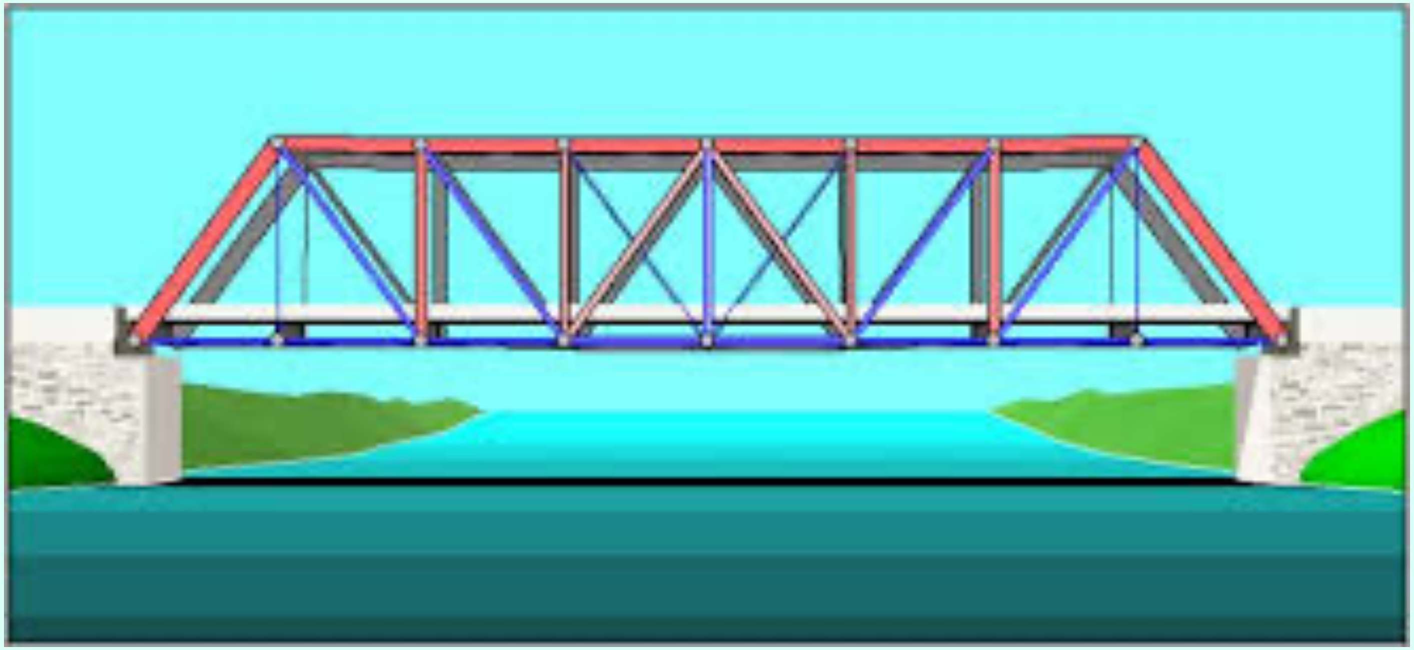
(3) Solve the examples in **Exercise Sheet #3** and check results using COMSOL.





3c. COMSOL Practical #2

Design of a truss bridge



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Next section…
(4) Beam Elements